MAC-CPTM Situations Project

Situation 55: Multiplication of Complex Numbers

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Prompt

The teacher of an Algebra III course notices that her students are having difficulty understanding some of the differences between the multiplication of real numbers, on the one hand, and the multiplication of complex numbers, on the other. The teacher wants her students to have a feasible way of thinking about the multiplication of complex numbers. In particular, she wants them to see these new numbers as having properties that are different from those of the real numbers.

Commentary

Foci 1 and 2 develop properties of i related to multiplication that help to distinguish its properties from those of the real numbers. Focus 1 examines how the property of orientation is changed when i is used in multiplication as opposed to when 1 is used. Focus 2 develops a definition of multiplication by i. These two foci establish two new characteristics of i as a number. Focus 3 examines how i can be considered a special complex number and extends to all complex numbers the definition of multiplication given in Focus 2. Further, Focus 3 connects the definition to rotation matrices and to the matrix representation of complex numbers. Finally, Focus 4 develops a different definition of complex multiplication that can be proved equivalent to the definition in Focus 3. The overall goal of the foci is to help establish images of multiplication for complex numbers that can subsequently be used in deriving some of the basic properties of those numbers. The approach in these foci is primarily geometric so as to help achieve the goal of establishing these numbers as quantities. The usual algebraic definitions can be derived from the geometric definitions.

Mathematical Foci

Mathematical Focus 1

Multiplication by i *can be interpreted in terms of oriented areas.*

We can think of imaginary numbers as relating to orientation. To this end, we may stipulate some conventions about orientation of a motion. The first is that motion to the right or up from a reference value will be considered motion in the positive direction. In contrast, motion to the left or down from a reference value will be considered motion in the negative direction. When two linear dimensions produce an area, movement will take place first in the horizontal direction and then in the vertical direction. Following this convention, area in the first quadrant will produce motion in the counterclockwise direction, which we take as the positive direction, so that the clockwise direction is negative. Note that the orientation of the area is established independently of the linear orientation. That is, we could have selected motion in the counterclockwise direction as negative. See the GSP animation of unit squares in the four quadrants.

Consider taking two unit lengths to produce a unit area. Because we can choose the orientation of the area independently of the orientation of the unit vectors, we now choose a counterclockwise direction as orienting the area negatively and a clockwise direction as orienting the area positively. We have not changed the magnitude of the area but only its orientation. The area still has a magnitude of 1, but its orientation has changed. The area is -1.

We introduce some symbolism to represent the lengths of the sides. This symbolism cannot coincide with the symbolism used when the orientation was positive, because we want to indicate that we have created something new. We call each side length *i,* and we see that $i^2 = -1$. (For these newly oriented areas, you many want to look at the GSP file again.) It is worth considering what happens in each quadrant. For instance, in Quadrant 2, the point moves around the area in the clockwise direction, which is positive, so $-i \bullet i = 1$ (as expected).

This model of an imaginary number can be used to motivate a property of imaginary numbers that differs from that of real numbers; namely, when we have two positively or two negatively oriented numbers, multiplication produces a negatively oriented number. Similarly, if we have one negatively and one positively oriented number, multiplication produces a positively oriented number. We can use this model to derive the common definition that $i = \sqrt{-1}$ by taking the square root of both sides of the equation $i^2 = -1$.

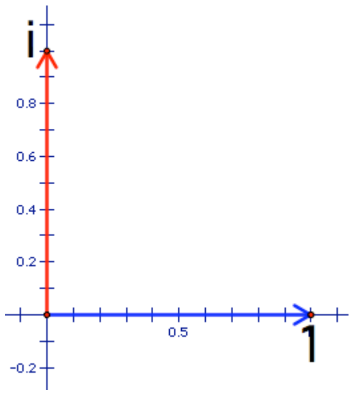
For more discussion of the orientation of real numbers, see Situation 27. With this basic model of imaginary numbers, see the end of Situation 50 for a discussion of how imaginary solutions could function in the context of factoring quadratics using an area model.

Mathematical Focus 2

Multiplication by i *can be interpreted as a vector rotation.*

Now that we have a basic model for imaginary numbers, we might want to develop a model that focuses on multiplicative composition. We first consider multiplicative composition of *i* with a unit vector. We can define our multiplicative operation to be a rotation of 90 degrees in the counterclockwise direction. We can make sense of this definition by thinking of multiplication by -1 as a counterclockwise rotation about the origin by 180 degrees. If we consider $-1 = i^2$, then it is consistent to define multiplication by *i* to be a rotation by half as many degrees as multiplication by -1 or i^2.

In defining multiplication this way, we have to be clear that we are constructing a new number system using an operator model. That is, *i* is operating on the unit vector. The vector 1 is rotated by 90 degrees counterclockwise to create the vector *i* on the y-axis.

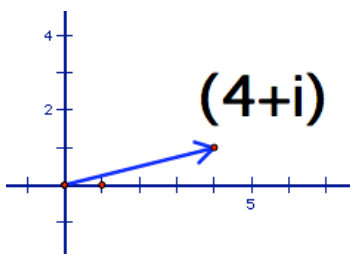


If we multiply the vector *i* by itself, we get i^2, which we know from Focus 1 is the same as -1. If we perform the multiplication by *i* two more times, we get $i^3 = -i$ and $i^4 = 1$ as expected. In this context, we can check that products are associative, commutative, and distributive in light of the geometric operations being used. Here again, we have a new property of an imaginary number that helps us to differentiate it from that of real numbers.

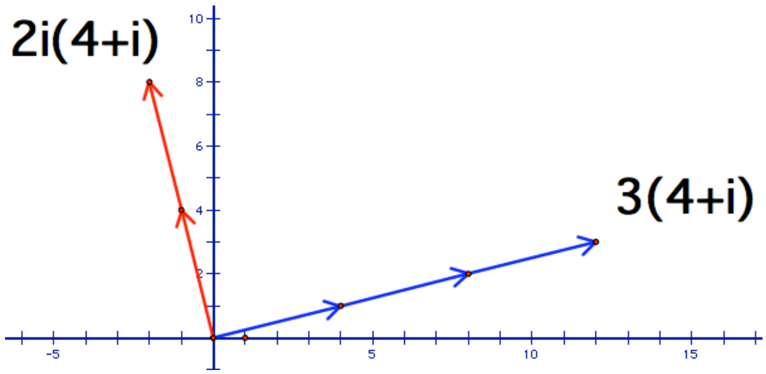
Mathematical Focus 3

More generally, multiplication by a *complex number can be interpreted as a vector rotation plus a dilation.*

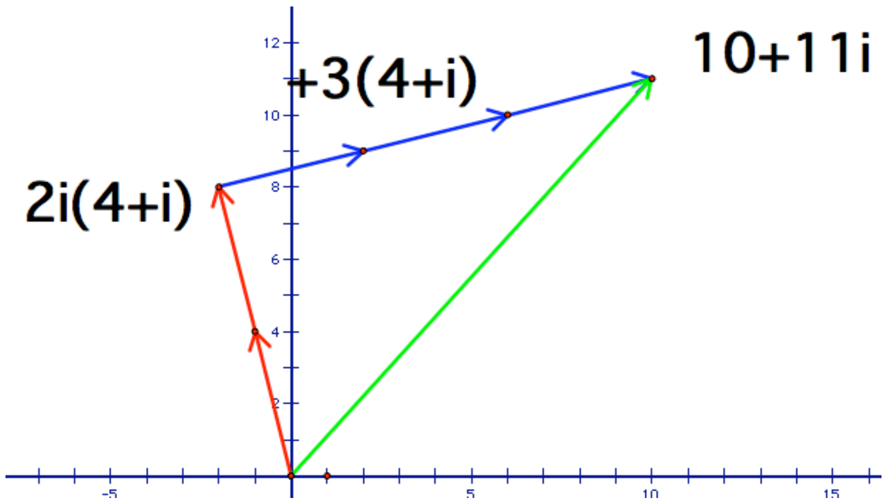
Let us test our notion of multiplication with some complex numbers that are not imaginary. For example, multiply (2*i* + 3) by (4 + *i*). The vector (4 + *i*) is depicted below:



Using the definition of multiplication by an imaginary number, when we multiply the vector (4 + *i*) by 2*i,* we rotate it 90 degrees counterclockwise and dilate it by 2, shown below as the red vector. When we multiply the vector (4 + *i*) by 3, we simply dilate it by 3, shown below as the blue vector.



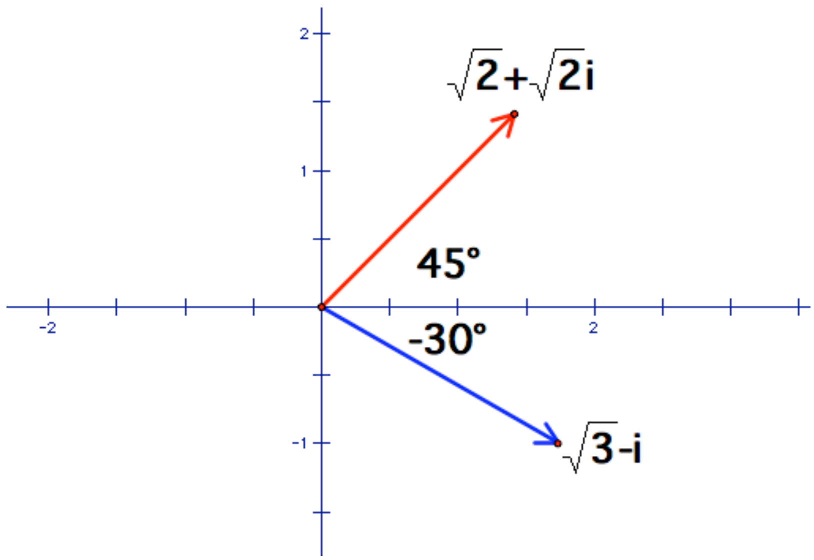
Finally, we want to add the blue and red vectors. The resulting product vector is shown below in green. One can use geometry to derive the usual algebraic definition of multiplication of complex numbers.



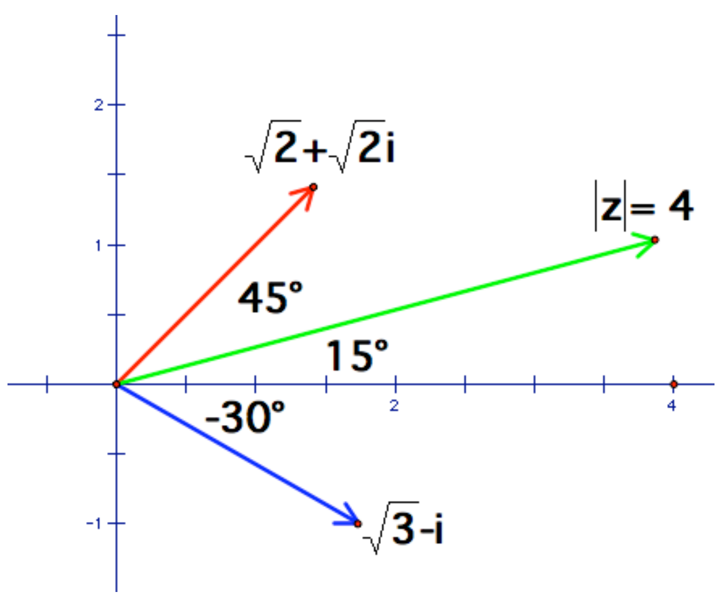
Also, it is now possible to coordinate the definition of multiplication by a complex number, the matrix representation of a complex number, and the definition of a rotation matrix. A complex number is sometimes represented as a matrix in the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, and a rotation matrix is defined as $\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$, where ϑ is the angle of rotation in the counterclockwise direction. We see that *bi* is represented by a matrix of the form $\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which we can interpret in terms of our rotation matrix as $\cos\vartheta = 0$ and $\sin\vartheta = 1$, giving a rotation of $\vartheta = 90^\circ$ followed by a dilation by the scale factor *b*. This result is consistent with our earlier definition. If we look at multiplication by the real number *a*, $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so that $\cos\vartheta = 1$ and $\sin\vartheta = 0$, and $\vartheta = 0°$ followed by a dilation by the scale factor *a*. Again, this result is consistent with our notion of multiplication by a real number.

Mathematical Focus 4
Multiplication by a *complex number can also be interpreted using a polar representation.*

We consider one final way of looking at complex multiplication. The previous model was essentially an operator model in which one complex number operated on the other. We now consider a model of multiplication in which the two vectors play an equal role. We define the vector that results from the multiplication of two complex numbers to be a vector whose length is the product of the lengths of the two complex numbers (their moduli) and which is located at the sum of their angles (their arguments). This definition is complicated but easily seen in a figure. Below is a depiction of the complex numbers $\sqrt{2}+\sqrt{2}i$ and $\sqrt{3}-i$:



To find their product, we calculate that the length of each is 2 (using the Pythagorean Theorem), and find the sum of the angles to be 15 degrees. Therefore, the product of the two numbers is a vector of length 4 at 15 degrees (see below):



We can then use some basic trigonometry to find the components of the product. The real component can be found by solving the equation $\cos 15° = \frac{x}{4}$, and the complex component by solving the equation $\sin 15° = \frac{y}{4}$. The resulting product is approximately $3.86 + 1.04i$. At this point it might be worth comparing the geometric definition with the standard algebraic definition, which yields $(\sqrt{6} + \sqrt{2}) + (\sqrt{6} - \sqrt{2})i$.

This definition of multiplication can be derived from the polar representation of a complex number $z = |z|(\cos\vartheta + i\sin\vartheta)$, or similarly by using rotation matrices. In other words, using polar coordinates, one multiplies two complex numbers by multiplying their moduli and adding their arguments. This definition is useful for representing features of complex numbers such as conjugates, inverses, and exponentiation (e.g., a complex number squared is the square of the length of the complex number located at twice the angle of the number being squared). A similar-triangles argument can be used to establish the equivalence of this definition with the one in Focus 3.